THE EFFECTS OF SIMPLE BOUNDARY IRREGULARITIES ON ACOUSTICAL MODE PROPAGATION IN DUCTS WITH PRESSURE RELEASE BOUNDARIES

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THESIS

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Effects of boundary irregularities on acoustical mode propagation in ducts, with pressure release walls are theoretically studied. Several particular wall irregularities are explored: Sinusoidal, δ-function and spatially decaying wall perturbations. Strong effects (resonances) appear in the case of the sinusoidal wall. In the other two cases, the resonances are not present, but there are still traveling disturbances whose effects can be important at great distances from the wall irregularity.

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x, z	rectangular coordinates
ℓ_z	transverse dimension of waveduct
k	acoustic wave number
ε	perturbation constant
ω	angular frequency of acoustic signal
ω_c	cut-off frequency
Φ	acoustic velocity potential
t	time
γ	spatial frequency
Ω	angular frequency of surface wave
s	transformed variable
m_c	critical value of m

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I. INTRODUCTION

The theory of acoustic wave propagation in bounded media (waveguides and waveducts) has been treated with a certain intensity in the last twenty-five years. It is possible to classify the problems arising in these studies in two categories. One category involves the effects on acoustic propagation of small inhomogeneities and the other, the effects on the acoustic field of irregularities on the boundaries of the waveguide or waveduct.

These two factors are the most important concerning the acoustic wave propagation and M.A. Isakovich [4] in 1956 said that "such problems, besides their theoretical interest are also important from a practical point of view; thus, for example, in super-long-range waveduct propagation of sound in a laminarly-inhomogeneous medium the effect of even small inhomogeneities or corrugation of the boundaries can produce a very great effect upon the range of propagation". M.A. Isakovich studied propagation in a waveduct with small inhomogeneities and in a waveduct with one corrugated wall, using the method of small perturbations. This was a natural extension of the problem of scattering of sound from one single rough wall.

I.C. Samuels [5], in 1958, studied the problem of propagation of acoustic and electromagnetic waves in ducts with two corrugated rigid walls. He found, in the case of

harmonic roughness, "that waves are not scattered upstream if the frequency of the transmitted signal exceeds certain critical values related to the natural modes of the waveguide and the wavelengths of the wall roughness."

C.S. Clay [6], in 1963 presented a paper where the effect of a slightly irregular boundary on waveguide propagation was studied. The results were checked by an experiment in shallow water and he found that the effects of small roughness on the surface are to increase the attenuation as a function of range and decrease the coherence of acoustical propagation.

In more recent years (1971), R.H. Ebert [7] calculated the fluctuation of sound due to influence of a standing gravity wave in a waveguide. The results showed a resonance if the wavelength of the surface wave was one-half of the acoustic wavelength. In an experiment realized in the Naval Postgraduate School he verified the existence of that resonance.

R.F. Salant [8], in 1972, stated that the research and study of the theory of sound propagation in waveguides could be and had been applied to several practical problems and gave as examples: transmission of noise through ventilating ducts, jet-engine inlets and other ducting systems. He considers propagation in a waveguide having two sinusoidal walls having a difference of phase, as opposed by earlier work by J.C. Samuels [5], whose waveguide had the sinusoidal walls in phase.

Salant concluded that the phase difference between the walls, the wall wave number and the mode number in the waveguide affect strongly the disturbance generated by the walls.

He found that the disturbances when odd modes are propagating in the waveguide are extremely different from the ones formed when even modes are propagating.

A. H. Nayfek [9] in 1974 treated again the acoustic propagation in ducts with sinusoidal walls using the method of multiple scales, in contrast with previous works where the small perturbations method was used. For travelling waves, resonance occurs wherever the wall wave number is equal to the difference of the wave number of any two duct acoustic modes.

N. E. Davis and M. W. Kenyon [1], in December 1974 studied the effects of travelling sinusoidal surface waves in a waveguide, verifying that resonances occur under certain conditions relating wavelengths of surface and acoustic waves. A waveguide with pressure release walls was built and the presence of such resonances verified.

In this work, a simple theory is presented for solutions of the acoustic velocity potential in irregular ducts with pressure release walls using the Fourier transform of the functions describing the irregularity of the wall.

II. PROBLEM DESCRIPTION

A. GENERAL FORMALISM

Consider the existence of a waveduct, with transverse dimension ℓ_z. Let the walls be pressure release and smooth, except for the upper wall that will be irregular. This waveduct will be excited by a monofrequency sound source, and is assumed to be of infinite length in the x direction.

For the case of perfectly smooth pressure release walls, a solution for the wave equation (velocity potential) is, taking for simplicity an amplitude of one,

$$\Phi = \sin(k_z z) \cos(\omega t - k_x x) \ . \qquad (2.1)$$

This represents a wave travelling in the positive x direction. In this equation, k_x and k_z must obey the relations:

$$\begin{aligned} k_x &= [(\tfrac{\omega}{c})^2 - k_z^2]^{1/2} \\ k_z &= \frac{n\pi}{\ell_z} \ , \quad n = 1,2,\ldots \end{aligned} \qquad (2.2)$$

Assume that the upper wall of the waveduct is irregular and located at

$$z = \ell_z + \varepsilon\, f(x) \qquad (2.3)$$

where f(x) is the function that describes the disturbance. Make $\varepsilon << 1$, so that the amplitude of the disturbance will be small compared with the dimension ℓ_z of the waveduct.

Assume a solution for this case

$$\Phi = \Phi_0 + \varepsilon\, \Phi_1 + \varepsilon^2\, \Phi_2 + \ldots + \varepsilon^n\, \Phi_n + \ldots \qquad (2.4)$$

In the above expression, Φ_0 is as given in equation (2.1), representing the solution for the undisturbed waveduct.

Since the walls are pressure release, the boundary conditions are:

$$\Phi = 0 \quad \text{at} \quad \begin{array}{l} z = 0 \\ z = \ell_z + \varepsilon\, f(x) \end{array}$$

$$\Phi_0 = 0 \quad \text{at} \quad \begin{array}{l} z = 0 \\ z = \ell_z \end{array} \quad . \qquad (2.5)$$

Writing Φ as a Taylor series expansion about the point $z = \ell_z$:

$$\Phi[\ell_z + \varepsilon f(x)] = \Phi_0(\ell_z) + \varepsilon[\Phi_1 + f(x) \frac{\partial \Phi_0}{\partial z}]_{\ell_z}$$

$$+ \varepsilon^2[\Phi_2 + f(x) \frac{\partial \Phi_1}{\partial z} + \frac{1}{2} f^2(x) \frac{\partial^2 \Phi_0}{\partial z^2}]_{\ell_z}$$

$$+ \ldots$$

$$+ \varepsilon^n [\sum_{j=0}^{n} \frac{f^j(x)}{j!} \frac{\partial^j \Phi_{n-j}}{\partial z^j}]_{\ell_z}$$

$$+ \ldots \qquad (2.6)$$

With the imposition of boundary conditions, Eq. (2.5) then yields the following set of equations:

$$\Phi_1\Big|_{\ell_z} = - \frac{\partial \Phi_0}{\partial z}\Big|_{\ell_z} \cdot f(x)$$

$$\Phi_2\Big|_{\ell_z} = - \frac{\partial \Phi_1}{\partial z}\Big|_{\ell_z} \cdot f(x) - \frac{1}{2!} \frac{\partial^2 \Phi_0}{\partial z^2}\Big|_{\ell_z} \cdot f^2(x)$$

$$\Phi_n\Big|_{\ell_z} = - \sum_{j=1}^{n} \frac{f^j(x)}{j!} \frac{\partial^j \Phi_{n-j}}{\partial z^j} \quad . \qquad (2.7)$$

In the assumed solution, Φ and Φ_n must satisfy the homogeneous, linear wave equations

$$\begin{aligned} \Box^2\Phi &= (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \Phi = 0 \\ \Box^2\Phi_n &= (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \Phi_n = 0 \quad . \end{aligned} \qquad (2.8)$$

Substitution of (2.1) into the first equation of (2.7) gives

$$\Phi_1\Big|_{\ell_z} = -k_z \cos(k_z z) \cos(\omega t - k_x x)\Big|_{\ell_z} \cdot f(x)$$

$$= -k_z \cos(n\pi) \cos(\omega t - k_x x) f(x) \quad . \qquad (2.9)$$

Equation (2.9) may be generalized by making it complex.

$$\underline{\Phi}_1\Big|_{\ell_z} = -f(x)\, k_z \cos(n\pi)\, e^{i(\omega t - k_x x)} \qquad (2.10)$$

such that

$$\mathrm{Re}\{\underline{\Phi}_1\Big|_{\ell_z}\} = \Phi_1\Big|_{\ell_z} \quad . \qquad (2.11)$$

The solution will then become

$$\Phi_1 = \mathrm{Re}\{\underline{\Phi}_1\} \quad . \qquad (2.12)$$

Define:

$$\underline{F}(x) = f(x)\, e^{-ik_x x} \quad . \qquad (2.13)$$

The insertion of definition (2.13) into Eq. (2.10) produces

$$\Phi_1\Big|_{\ell_z} = -e^{i\omega t} k_z \cos(n\pi)\, \underline{F}(x) \quad . \qquad (2.14)$$

By use of the definition of the Fourier transform of $\underline{F}(x)$

$$\underline{G}(s) = \int_{-\infty}^{+\infty} \underline{F}(x)\, e^{-ixs}\, dx$$

and its inverse (2.15)

$$\underline{F}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{G}(s)\, e^{ixs}\, dx\ ,$$

the boundary condition, Eq. (2.14), becomes:

$$\underline{\Phi}_1 \Big|_{\ell_z} = -e^{i\omega t}\, k_z \cos(n\pi)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{G}(s)\, e^{ixs}\, ds\ . \tag{2.16}$$

Use as a trial solution:

$$\underline{\Phi}_1 = -e^{i\omega t}\, k_z \cos(n\pi)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(Kz)}{\sin(K\ell_z)}\, \underline{G}(s)\, e^{ixs}\, ds\ . \tag{2.17}$$

Since the above equation must satisfy Eq. (2.8) [the homogeneous, linear wave equation], the quantity K must obey the relationship

$$K = [(\frac{\omega}{c})^2 - s^2]^{1/2}\ . \tag{2.18}$$

B. SINUSOIDAL UPPER WALL

Let the upper wall of the waveduct be perturbed in the form of a cosinusoide of amplitude $\varepsilon \ell_z$ where $\varepsilon << 1$. (See Figs. 1 and 2) The upper boundary condition is

$$\Phi = 0 \quad \text{at} \quad z = \ell_z(1 + \varepsilon \cos \gamma x) \quad .$$

Thus f(x) is specified as

$$f(x) = \ell_z \cos(\gamma x) \quad . \tag{2.19}$$

Inserting Eq. (2.19) into Eq. (2.13)

$$\underline{F}(x) = \ell_z \cos(\gamma x)\; e^{-ik_x x}$$

$$= \ell_z \left[\frac{e^{i\gamma x} + e^{-i\gamma x}}{2} \right] \cdot e^{-ik_x x} \quad . \tag{2.20}$$

The Fourier transform of (2.20) is:

$$\underline{G}(x) = \frac{\ell_z}{2} \int_{-\infty}^{\infty} [e^{-i(k_x-\gamma)x} + e^{-i(k_x+\gamma)x}]\, e^{-ixs}\, ds$$

$$= \pi \ell_z \;[\delta(s+k_x-\gamma) + \delta(s+k_x+\gamma)] \tag{2.21}$$

and the solution will be:

$$\underline{\Phi}_1 = -e^{i\omega t}\, k_z \cos(n\pi)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin Kz}{\sin K\ell_z} \left\{ \pi \ell_z [\delta(s+k_x-\gamma) + \delta(s+k_x+\gamma)] \right\} \cdot$$

$$\cdot\; e^{ixs}\, ds \quad . \tag{2.22}$$

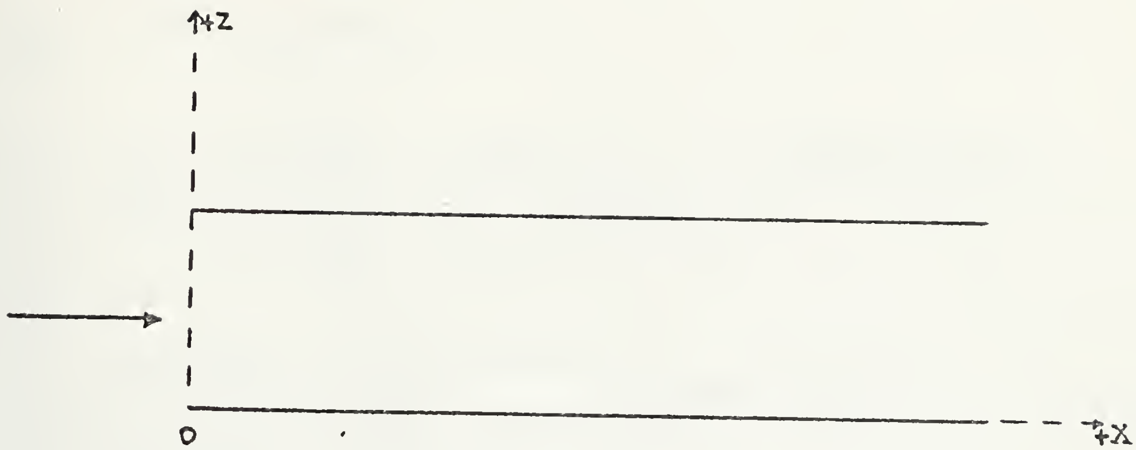

FIGURE 1. The Unperturbed Duct

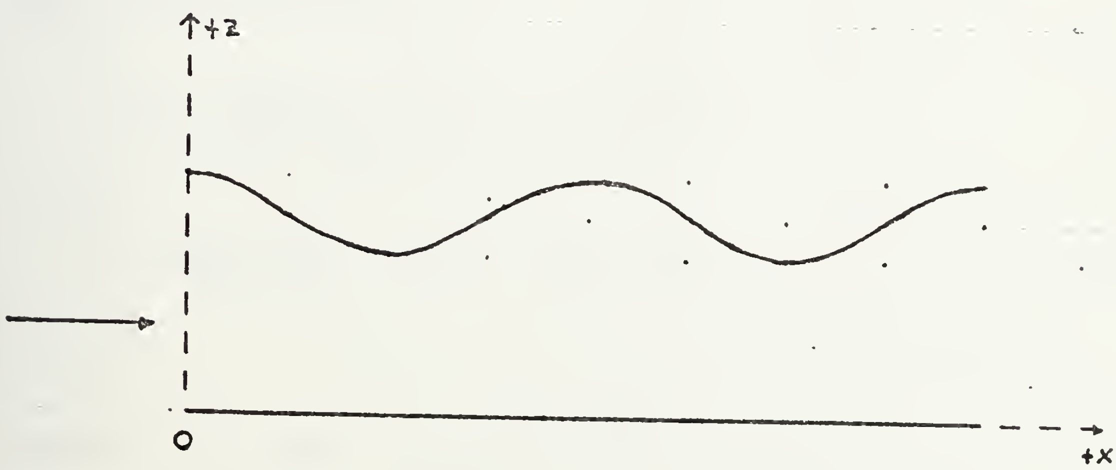

FIGURE 2. The Perturbed Duct

Using the properties of the δ-function, the integral is readily evaluated so that

$$\underline{\Phi}_1 = -e^{i\omega t}\, k_z\,{}_z \cos n\pi \,\cdot$$

$$\cdot\ \frac{1}{2}\left\{ \frac{\sin[(\frac{\omega}{c})^2 - (k_x+\gamma)^2]^{1/2}\, z}{\sin[(\frac{\omega}{c})^2 - (k_x+\gamma)^2]^{1/2}\, \ell_z}\, e^{i[\omega t-(k_x+\gamma)x]} + \right.$$

$$\left. + \frac{\sin[(\frac{\omega}{c})^2 - (k_x-\gamma)^2]^{1/2}\, z}{\sin[(\frac{\omega}{c})^2 - (k_x-\gamma)^2]^{1/2}\, \ell_z}\, e^{i[\omega t-(k_x-\gamma)x]} \right\} . \qquad (2.23)$$

Since $\Phi_1 = \mathrm{Re}\{\underline{\Phi}_1\}$, the first order correction is

$$\underline{\Phi}_1 = -k_z \ell_z \cos(n\pi) \,\cdot$$

$$\cdot\ \frac{1}{2}\left\{ \frac{\sin K^+ z}{\sin K^- \ell_z} \cos[\omega t - (k_x+\gamma)x] + \right.$$

$$\left. + \frac{\sin K^- z}{\sin K^- \ell_z} \cos[\omega t - (k_x-\gamma)x] \right\} , \qquad (2.24)$$

where $K^{\pm} = [(\frac{\omega}{c})^2 - (k_x \pm \gamma)^2]^{1/2}$. (2.25)

1. <u>Validity and Confirmation of Solution</u>

The above solution is consistent with the results presented by N. E. Davis and M. W. Kenyon [1]. The waveguide, in that case is perturbed by a traveling surface wave of

amplitude A. The upper boundary condition is $\Phi = 0$ at $z = \ell_z + \varepsilon F(x,t)$, where

$$F(x,t) = -\ell_z \cos(\Omega t - \gamma x) \quad \text{and} \quad \varepsilon = \frac{A}{\ell_z} \ .$$

By making $\Omega = 0$ in Eq. (2.12) of [1] and discarding the functional dependence in the y-direction, the result agrees completely with Eq. (2.22).

2. Properties of the Solution

In Eq. (2.24), the propagation constant in the z-direction is

$$K^{\pm} = [(\frac{\omega}{c})^2 - (k_x \pm \gamma)^2]^{1/2} \ . \tag{2.26}$$

Since $(\frac{\omega}{c})^2 = k_x^2 + k_z^2$,

$$K^{\pm} = k_z \left[1 - \frac{(1 \pm 2 \frac{k_x}{\gamma})}{(k_z/\gamma)^2}\right]^{1/2} \ . \tag{2.27}$$

If $K^{\pm}$ is real and:

$$K^{\pm} \ell_z = m\pi \quad , \quad m=1,2,\ldots, \tag{2.28}$$

the conditions for a resonant situation exist. From Eq. (2.2), $k_z = \frac{n\pi}{\ell_z}$, $n=1,2,\ldots$. Inserting this into

Eq. (2.28) the following expression results, which states the relation necessary for a resonance:

$$n\left[1 - \frac{(1 \pm 2\, k_x/\gamma)}{(k_z/\gamma)^2}\right]^{1/2} = m$$

or

$$\left(\frac{m}{n}\right)^2 = 1 - \frac{(1 \pm 2\, k_x/\gamma)}{(k_z/\gamma)^2} \quad . \tag{2.29}$$

The resonance curves are shown in Figure 3 as a plot of k_x/γ versus $(k_z/\gamma)^2$. This is in accordance with Eq. (2.16) and Figure 2 of [1], and [3].

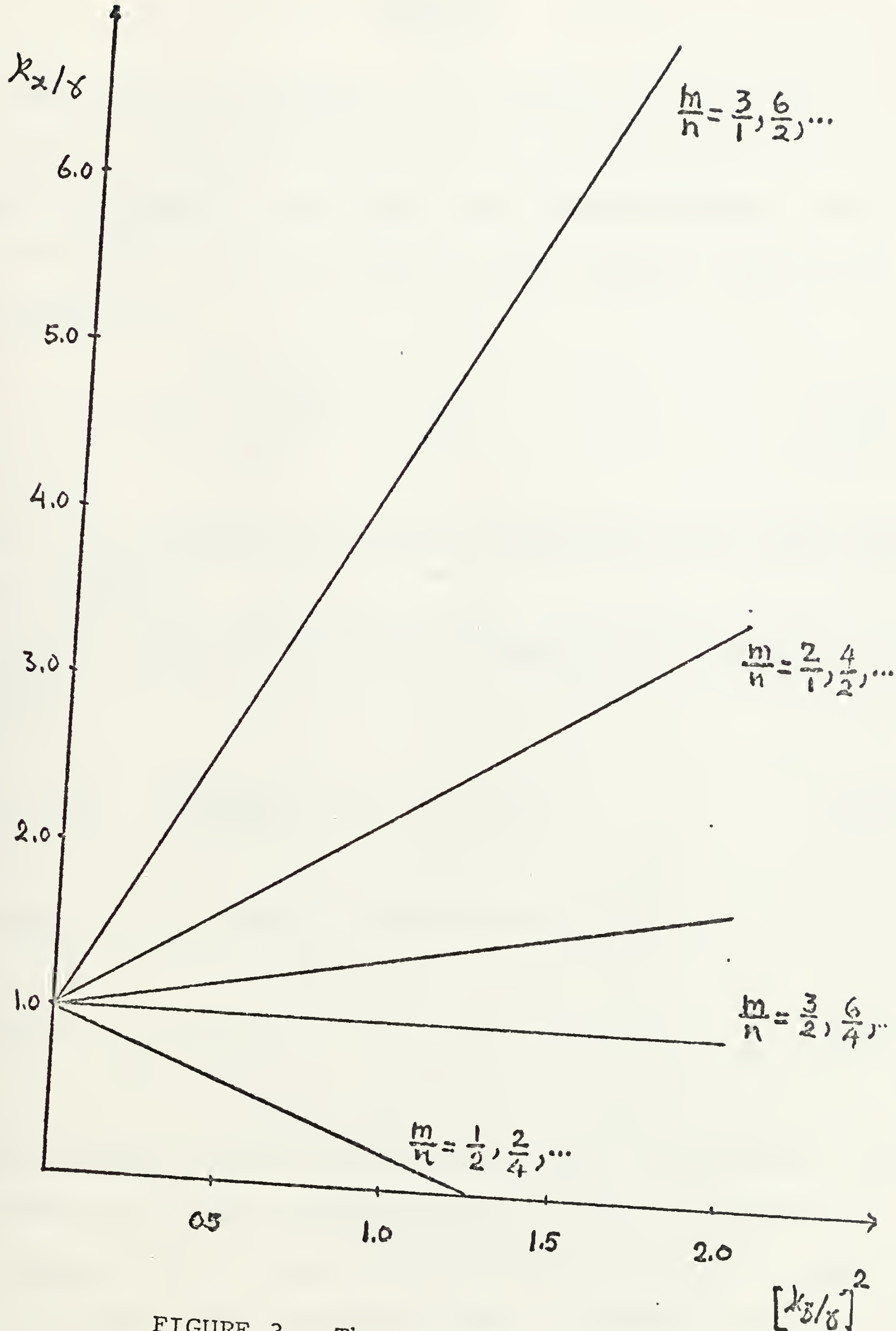

FIGURE 3. The Resonance Curves

III. ANALYSIS

Using spectral analysis it is possible to find the solution to the problem involving any special boundary condition related to the upper wall of the waveduct. The only thing that has to be known is the Fourier transform of the combination

$$f(x) \cdot e^{-ik_x x} \quad , \tag{3.1}$$

where f(x) represents the function describing the upper wall of the waveduct.

Knowing this, the problem is reduced to solve the integral

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin Kz}{\sin K\ell_z} \underline{G}(s)\, e^{ixs}\, ds \quad , \tag{3.2}$$

where $\underline{G}(s)$ is the Fourier transform of (3.1),

$$\underline{G}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i(k_x+s)x}\, dx \quad . \tag{3.3}$$

In the problem treated and solved before, the upper wall was described by a very simple function and the evaluation of the integral did not pose any special difficulties. But, if f(x) is a more complicated function, a careful analysis of (3.2) is necessary in order to solve it.

A. THE SOLUTION INTEGRAL

If s is generalized to a complex quantity (s is the variable of integration), a careful study of the behavior of the function under the integral sign in (3.2) in the complex s-plane is necessary.

The function to study is:

$$f(s) = \frac{\sin[(\frac{\omega}{c})^2 - s^2]^{1/2} z}{\sin[(\frac{\omega}{c})^2 - s^2]^{1/2} \ell_z} \underline{G}(s) e^{ixs} . \qquad (3.4)$$

1. The Poles

Assuming, for the moment, that $\underline{G}(s)$ does not have any poles, interest is focused in the ratio of sines in (3.4). Since [2]

$$\sin z = z \prod_{m=1}^{\infty} (1 - \frac{z^2}{m^2 \pi^2}) , \qquad (3.5)$$

f(s) can be written as

$$f(s) = \frac{[(\frac{\omega}{c})^2 - s^2]^{1/2} z \prod_{m=1}^{\infty} \left\{1 - \frac{[(\frac{\omega}{c})^2 - s^2] z^2}{m^2 \pi^2}\right\} \underline{G}(s) e^{ixs}}{[(\frac{\omega}{c})^2 - s^2]^{1/2} \ell_z \prod_{m=1}^{\infty} \left\{1 - \frac{[(\frac{\omega}{c})^2 - s^2] \ell_z^2}{m^2 \pi^2}\right\}} \qquad (3.6)$$

or simply

$$f(s) = \frac{z}{\ell_z} \prod_{m=1}^{\infty} \frac{\left\{m^2\pi^2 - [(\frac{\omega}{c})^2 - s^2]\ z^2\right\}}{\left\{m^2\pi^2 - [(\frac{\omega}{c})^2 - s^2]\ \ell_z^{\,2}\right\}}\ \underline{G}(s)\ e^{ixs}\ . \qquad (3.7)$$

There are no branch cuts, the poles of f(s) are simple and located at

$$s_m = \pm\ [(\frac{\omega}{c})^2 - (\frac{m\pi}{\ell_z})^2]^{1/2} \qquad (3.8)$$

$$m=1,2,...$$

It is possible to arrive at the same result by simply looking at the denominator of (3.4). The poles will be located at

$$[(\frac{\omega}{c})^2 - s^2]^{1/2}\ \ell_z = \pm\ m\pi \qquad (3.9)$$

$$m=1,2,...\ .$$

Manipulation reveals that Eq. (3.9) is equivalent to Eq. (3.8), but here it is not evident that the poles are simple, and that there are no branch cuts.

From Eqs. (2.2) it is known that

$$(\frac{\omega}{c})^2 = k_x^{\,2} + k_z^{\,2}$$

and

$$k_z = \frac{n\pi}{\ell_z} \quad ; \quad n=1,2,\ldots \quad .$$

Using this in Eq. (3.8), it can be shown that the location of the poles are at

$$s_m = \pm \left[k_x^2 + \left(\frac{\pi}{\ell_z}\right)^2 (n^2-m^2)\right]^{1/2} \tag{3.10}$$

$$n,m=1,2,\ldots \quad .$$

Looking at this expression, it can bee seen that the poles will be located in the real axis if

$$k_x^2 > \left(\frac{\pi}{\ell_z}\right)^2 (n^2-m^2) \tag{3.11}$$

$$n,m=1,2,\ldots \quad ,$$

and on the imaginary axis if the inequality is reversed.

Let us choose n=1 for all that follows. This means that only the lowest mode can propagate energy. It can be seen that as m increases the associated poles are found successively closer to the origin on the real axis until above some critical value, m_c, the poles lie on the imaginary axis with distance from the origin increasing with m. The first pole, s_1, will be located at

$$s_1 = \pm k_x \quad . \tag{3.12}$$

If it is assumed that $m_c = 3$, the fourth pole is already on the imaginary axis. In Table I and Figure 4 a situation like this is shown.

2. Effects of Absorption

If the effects of absorption are introduced, the wave equation that governs the propagation inside the duct is modified and becomes:

$$(\Box^2 - 2\frac{\alpha_n}{c}\frac{\partial}{\partial t})\,\Phi_n = 0 \tag{3.13}$$

$$n = 0,1,2,\ldots ,$$

where the α_n's are the coefficients of absorption responsible for losses. The α_n's can be assumed as having the same magnitude since all the Φ's have identical form, and the propagation constant in the x-direction will become complex.

It is known that the Φ's have the form

$$\sin(k_z z)\,e^{i[\omega t - \underline{k}_x x]} \tag{3.14}$$

Using this in Eq. (3.13), the value of $\underline{k}_x$ is

$$\underline{k}_x = k_x - i\alpha\frac{\omega}{ck_x} = k_x - i\beta$$

where (3.15)

$$\beta = \alpha\frac{\omega}{ck_x}$$

TABLE I

The Poles of $f(s)$

$n=1\ ,\ m=1$:	$s_1 = \pm\, k_x$
$n=1\ ,\ m=2$:	$s_2 = \pm\, [k_x{}^2 - 3(\frac{\pi}{\ell_z})^2]^{1/2}$
$n=1\ ,\ m=3$:	$s_3 = \pm\, [k_x{}^2 - 8(\frac{\pi}{\ell_z})^2]^{1/2}$
$n=1\ ,\ m=4$:	$s_4 = \pm\, [k_x{}^2 - 15(\frac{\pi}{\ell_z})^2]^{1/2}$
	$= \pm\, i\, [15(\frac{\pi}{\ell_z})^2 - k_x{}^2]^{1/2}$
$n=1\ ,\ m=5$:	$s_5 = \pm\, i\, [24(\frac{\pi}{\ell_z})^2 - k_x{}^2]^{1/2}$
$n=1\ ,\ m=6,7,\ldots$:	$s_m = \pm\, i\, [(m^2-1)(\frac{\pi}{\ell_z})^2 - k_x{}^2]^{1/2}$

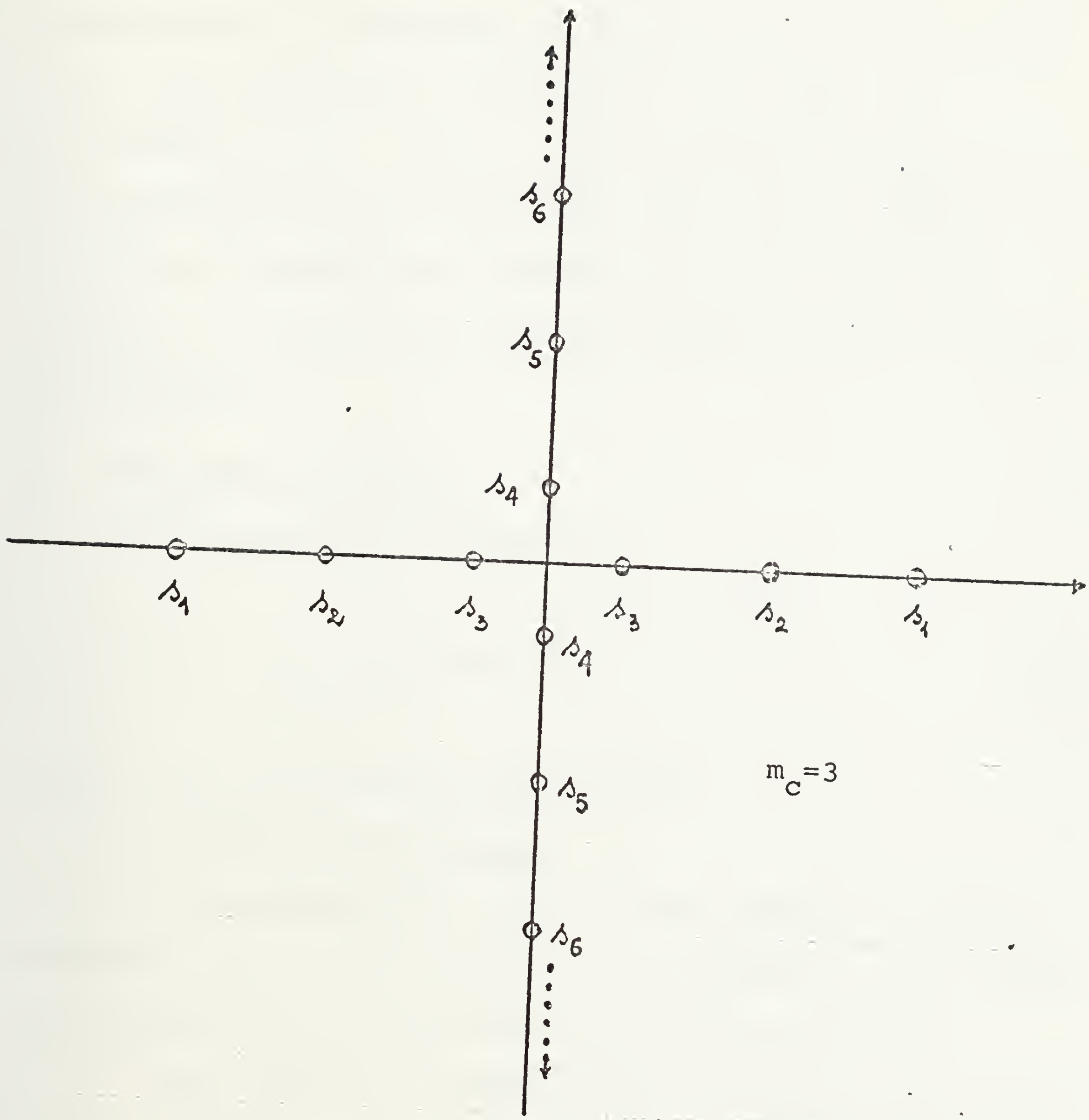

FIGURE 4. The Poles of f(s)

is the total absorption coefficient in the x-direction and will be considered very small.

It is necessary to know what is the effect of the introduction of absorption in the location of the poles of f(s), especially the ones located in the real axis. If s_1 is located at $s_1 = \pm \underline{k}_x$, it can be seen that the poles in the negative part of the real axis will be moved up and the poles located in the positive part will move down (Figure 5). We may now choose a contour in order to solve the integral.

B. THE CONTOUR INTEGRAL

The integral to be solved is

$$I = \int_{-\infty}^{\infty} f(s)\, ds \qquad (3.16)$$

where f(s) is the expression in (3.4).

1. Choice of the Contour

The integral in (3.16) is very similar to the one Isakovich used in his work [4]. It reduces to $2\pi i$ times the summation of the residues at the poles of f(s).

The contour to consider will consist of the segment [-R,+R] on the real axis and the upper semicircle C_R, as shown in Figure 5.

Therefore

$$I = \int_{-\infty}^{\infty} f(s)\, ds = \lim_{R\to\infty} \int_{-R}^{+R} f(s)\, ds = 2\pi i \sum_m \mathrm{Res}_m\, f(s) \; . \qquad (3.17)$$

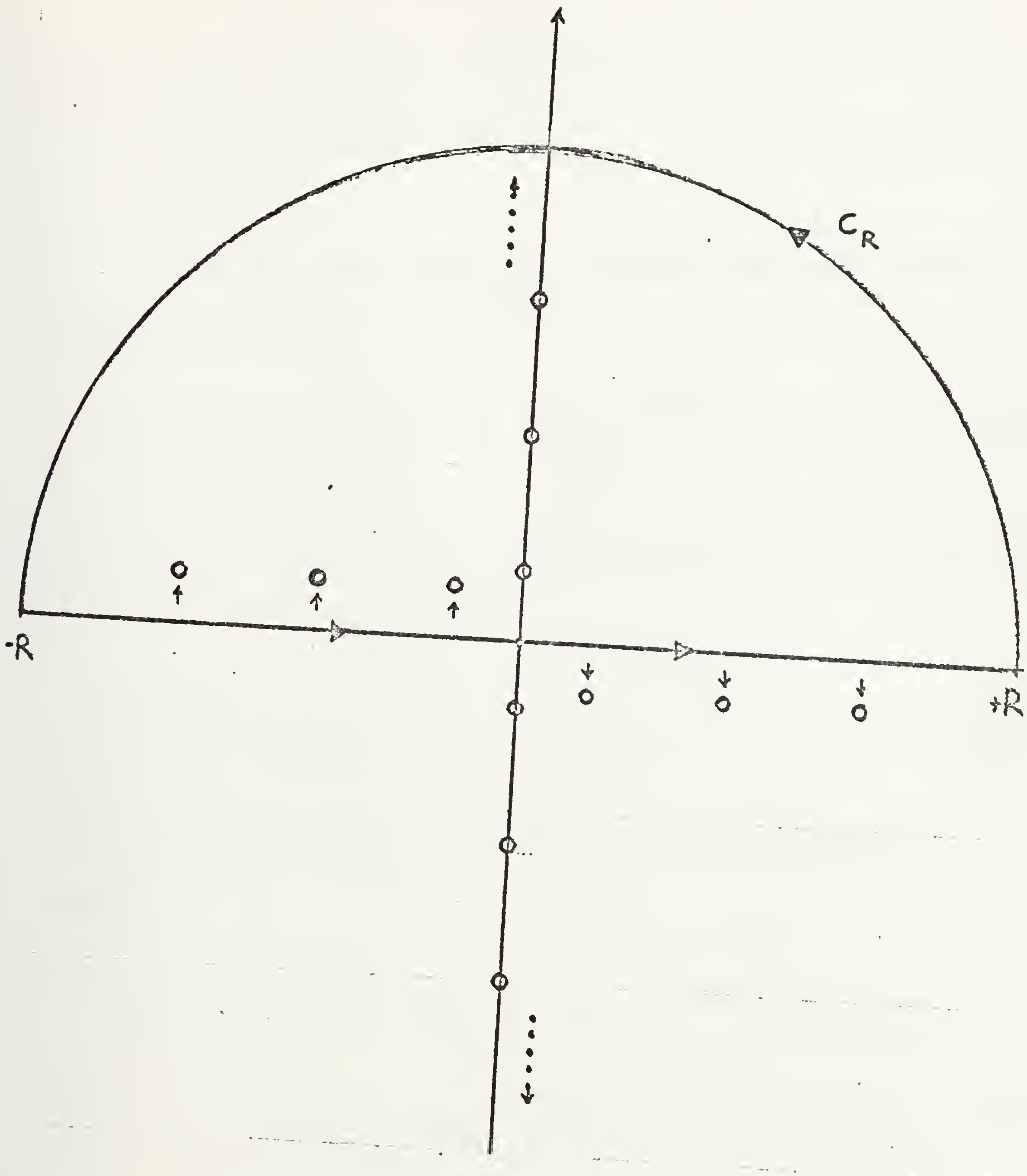

FIGURE 5. Effects of Absorption on the Location of the Poles of f(s). Contour.

Simplify Eq. (3.4) by defining H(s) as:

$$H(s) = \frac{\sin[(\frac{\omega}{c})^2 - s^2]^{1/2} \; z}{\sin[(\frac{\omega}{c})^2 - s^2]\ell_z} . \qquad (3.18)$$

For the case where $\underline{G}(s)$ does not have any poles the integral will be:

$$I = \int_{-\infty}^{\infty} f(s)\, ds = 2\pi i \sum_m \underline{G}(s_m)\, e^{ixs_m} \, \mathrm{Res}\{H(s_m)\} \qquad (3.19)$$

On the other hand, if the Fourier transform of $\underline{F}(x)$ has poles of its own, at $s = s_g$, where

$$\frac{1}{\underline{G}(s_g)} \to 0 \quad , \qquad (3.20)$$

the expression to evaluate the integral is modified as follows:

$$I = \int_{-\infty}^{\infty} f(s)\, ds = 2\pi i \sum_m \underline{G}(s_m)\, e^{ixs_m} \, \mathrm{Res}\{H(s_m)\}$$

$$+ 2\pi i \sum_g H(s_g)\, e^{ixs_g} \, \mathrm{Res}\{\underline{G}(s_g)\} \qquad (3.21)$$

In either case, the poles of interest will be the ones included within the contour chosen.

2. Definition of the Transform of $\underline{F}(x)$

In order always to be able to close the contour in the upper half plane, the definition of the Fourier transform of $\underline{F}(x)$ must be altered.

We define

$$\underline{G}(s) = \int_{-\infty}^{\infty} \underline{F}(x)\, e^{\mp isx}\, dx$$

with inverse

$$\underline{F}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{G}(s)\, e^{\pm ixs}\, ds \quad .$$

The upper sign of the exponent will represent the positive x-part of the waveduct with waves standing or propagating to the right and the lower sign will represent the negative x-part of the waveduct with waves standing or propagating to the left.

IV. SPECIAL CASES

In order to simplify the problem and apply this theory to some specific cases we shall impose some conditions on the problem. Since

$$\left(\frac{\omega}{c}\right)^2 = k_x^{\,2} + k_z^{\,2} \quad , \tag{4.1}$$

in the duct, the cut-off frequency will be

$$\omega_c = c\, k_z \quad . \tag{4.2}$$

The sound source that is going to excite the duct will have a monofrequency of

$$\omega = A\, c\, k_z \quad , \tag{4.3}$$

where

$$1 < A < 2 \tag{4.4}$$

to assure propagation only in the lowest modes. Under this condition and knowing that

$$k_z = \frac{\pi}{\ell_z} \quad , \tag{4.5}$$

(only lowest mode allowed in z, n=1) the location of the poles of f(s) will be at

$$s_m = \pm \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{\ell_z}\right)^2\right]^{1/2}$$

$$= \pm \frac{\pi}{\ell_z} \left[A^2 - m^2\right]^{1/2} \quad . \tag{4.5}$$

It can be seen that only one pole will be on the real axis (s_1), and that all others are already on the imaginary axis.

The general expression for the location of the poles of f(s), assuming $\underline{G}(s)$ has no poles is:

$$s_m = \pm\left[k_x^2 + \left(\frac{\pi}{\ell_z}\right)^2 (1 - m^2)\right]^{1/2} \tag{4.6}$$

and the m^{th} residue is

$$\mathrm{Res}_m = \frac{\sin\left\{\left[\left(\frac{\omega}{c}\right)^2 - s^2\right] z^2\right\}^{1/2} \cdot \underline{G}(s)\, e^{\pm ixs}}{\cos\left\{\left[\left(\frac{\omega}{c}\right)^2 - s^2\right] \ell_z^2\right\}^{1/2} \cdot \frac{1}{2}\left\{\left[\left(\frac{\omega}{c}\right)^2 - s^2\right] \ell_z^2\right\}^{-1/2} \cdot -2s\ell_z^2} \tag{4.7}$$

evaluated at $s = s_m$.

This will become, after manipulation,

$$\mathrm{Res}_m = (-1)^{m+1} \frac{mk_z}{s_m \ell_z} \sin \frac{m\pi}{\ell_z} z\, \underline{G}(s_m)\, e^{\pm ixs_m} \tag{4.8}$$

$$m = 1,2,\ldots \quad .$$

The effect on acoustic velocity potential due to the m^{th} pole is [using Eq. (2.17)]

$$e^{i\omega t} k_z \frac{1}{2\pi} \cdot 2\pi i \sum_m \mathrm{Res}_m \quad . \tag{4.9}$$

If $\underline{G}(s)$ has poles of its own, (4.9) will be modified and the effect on the velocity acoustic potential will be

$$e^{i\omega t} k_z \frac{1}{2\pi} \cdot 2\pi i \left[\sum_m \mathrm{Res}_m + \sum_g H(s_g)\, e^{\pm ixs_g} \mathrm{Res}_g \left\{ \{G(s_g)\} \right\} \right] . \tag{4.10}$$

A. THE δ-FUNCTION DISTURBANCE

Define $f(x)$, the function describing the wall disturbance as

$$f(x) = -\ell_z \; \delta(x) \quad . \tag{4.11}$$

Then

$$\underline{F}(x) = -\ell_z \; \delta(x) \; e^{-ik_x x} \tag{4.12}$$

and

$$\underline{G}(s) = -\ell_z \tag{4.13}$$

with no poles.

The poles of f(s) will be located according to Eq. (4.6). The first pole, the only one in the real axis will be at

$$s_1 = -k_x \tag{4.14}$$

and the second one, on the imaginary axis will be at

$$s_2 = + i \ [(\frac{\pi}{\ell_z})^2(2^2 - 1) - k_x^{\ 2}]^{1/2} \tag{4.15}$$

$$= + iP$$

where P is defined as

$$P = [3(\frac{\pi}{\ell_z})^2 - k_x^{\ 2}]^{1/2} \quad . \tag{4.16}$$

The overall situation is shown in Figures 6 and 7. The residue of f(s) at s_1, $Res_1(s_1)$ will be, using Eq. (4.8),

$$Res_1(s_1) = \frac{k_z}{k_x} \sin k_z z \ e^{\mp i k_x x} \tag{4.17}$$

The acoustic potential due to this residue will be calculated using (4.9), and

$$\underline{\Phi}_1\left\{Res_1(s_1)\right\} = \frac{k_z^{\ 2}}{k_x} \sin k_z z \ i \ e^{i(\omega t \mp k_x x)} \quad . \tag{4.18}$$

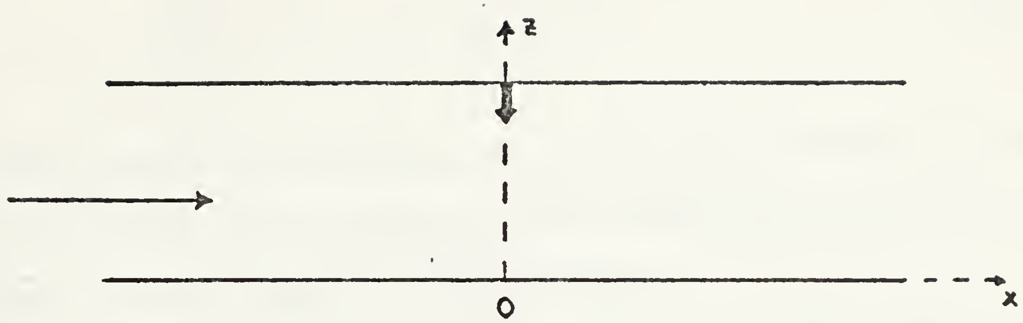

FIGURE 6. The δ-Function Disturbance

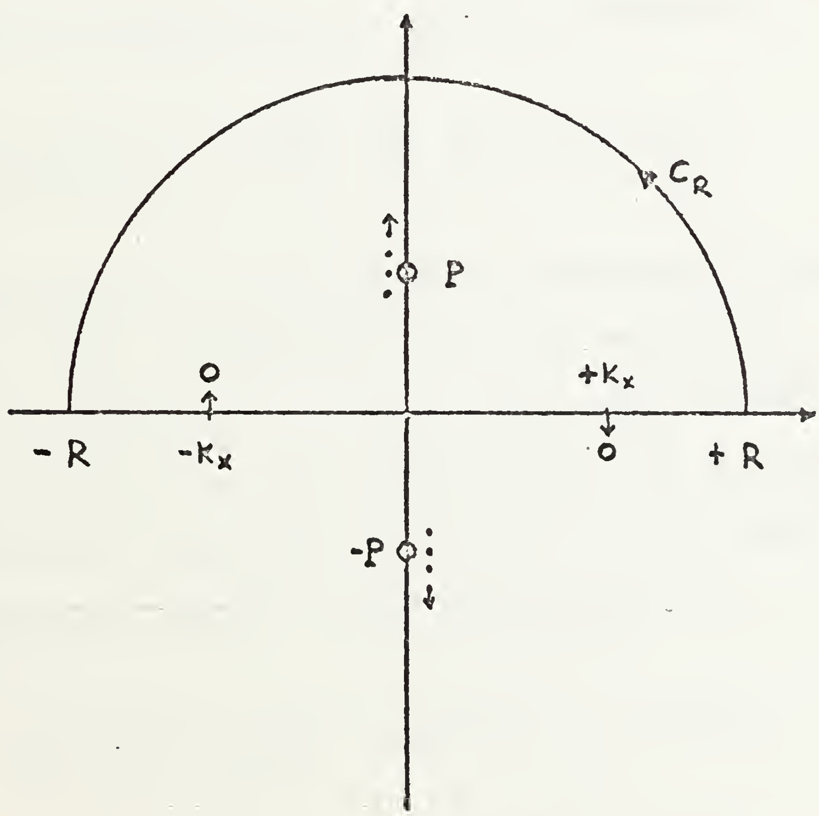

FIGURE 7. The Poles of Interest

Since $\phi_1 = \mathrm{Re}\{\underline{\Phi}_1\}$, in this case

$$\phi_1 = - \frac{k_z^{\,2}}{k_x} \sin k_z z \sin(\omega t \mp k_x x) \tag{4.19}$$

The above equation represents two scattered traveling waves emanating from x=0, one going in the positive x-direction and the other in the negative x-direction.

The residue of f(s) at s_2, $\mathrm{Res}_2(s_2)$, will be, using Eq. (4.8),

$$\mathrm{Res}_2(s_2) = -i \frac{2k_z}{P} \sin 2k_z z \; e^{\mp xP} \tag{4.20}$$

with a corresponding acoustic velocity potential of

$$\underline{\Phi}_1[\mathrm{Res}_2(s_2)] = \frac{2k_z^{\,2}}{P} \sin 2k_z z \; e^{i\omega t} \; e^{\mp xP} \quad . \tag{4.21}$$

This represents two disturbances in the acoustic field, decaying in the $\pm$ x-direction. It can be seen that far away from the wall disturbance their effect will be negligible.

Very close to the wall disturbance, x=0, the effects will be considerable and depend on the size of P_m, diminishing as P_m increases with m. P_m is defined as

$$P_m = [(m^2-1)(\frac{\pi}{\ell_z})^2 - k_x^{\,2}]^{1/2} \quad , \qquad m=3,4,\ldots \quad .$$

However, even at great distances from the irregularities in the wall the acoustic field is affected by $\phi_1[\mathrm{Res}_1(s_1)]$.

The total field, discarding the effects of decaying disturbances as in (4.21) is:

$$\Phi = \Phi_0 + \varepsilon\,\Phi_1$$

$$= \sin k_z z[\cos(\omega t - k_x x) - \varepsilon \frac{k_z^{\;2}}{k_x} \sin(\omega t \mp k_x x)] \tag{4.22}$$

Note that the traveling perturbation is 90° out of phase with the initial wave.

B. THE DECAYING WALL

Define f(x), the function describing the wall disturbance as

$$f(x) = -\ell_z\, e^{-|ax|} \tag{4.23}$$

Then

$$\underline{F}(x) = -\ell_z\, e^{-|ax|}\, e^{-ik_x x} \tag{4.24}$$

and

$$\underline{G}(s) = -\frac{2\ell_z}{|a|[1 + (\frac{s}{a} + k_x)^2]} \tag{4.25}$$

with poles located at

$$s_g = a(-k_x \pm i) \tag{4.26}$$

The poles of interest of f(s) will be located at

$$s_1 = -k_x$$

$$s_2 = +iP \qquad (4.27)$$

$$s_g = a(-k_x + i)$$

The overall situation is shown in Figures 8 and 9. Since $\underline{G}(s)$ has poles, Eq. (3.21) must be used to evaluate the integral.

The residue of f(s) at s_1, $\mathrm{Res}_1(s_1)$ will be

$$\mathrm{Res}_1(s_1) = \frac{2}{a}\frac{k_z}{k_x} \sin k_z z \frac{a^2}{a^2 + k_x{}^2(a-1)^2} e^{ik_x x} \qquad (4.28)$$

with a corresponding acoustic velocity potential of:

$$\underline{\Phi}_1[\mathrm{Res}_1(s_1)] = i\ 2\frac{k_z{}^2}{k_x}\frac{a}{a^2 + k_x{}^2(a-1)^2} \sin k_z z\ e^{i(\omega t \mp k_x x)} \qquad (4.29)$$

representing two traveling waves emanating from x=0, going in opposite directions, and

$$\Phi_1 = \mathrm{Re}\{\underline{\Phi}_1\} = -\frac{2k_z{}^2}{k_x}\frac{a}{a^2 + k_x{}^2(a-1)^2} \sin k_z z \sin(\omega t \mp k_x x) \qquad (4.30)$$

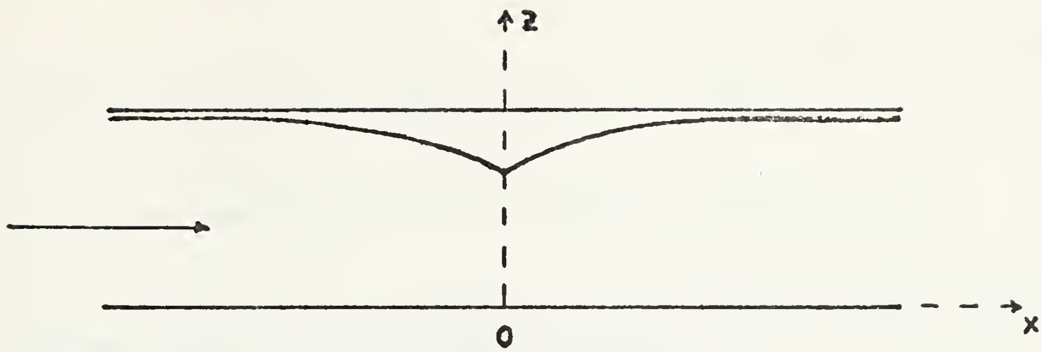

FIGURE 8. The Decaying Wall

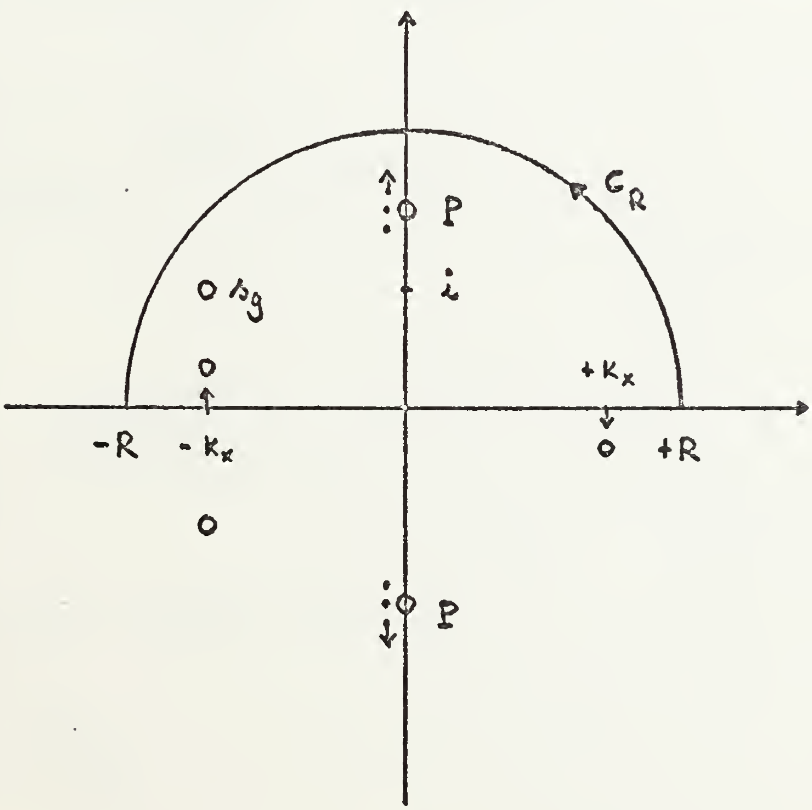

FIGURE 9. The Poles of Interest

The residue of f(s) at s_2, $Res_2(s_2)$ will be:

$$Res_2(s_2) = \frac{4k_z}{iPa[1 + (\frac{iP}{a} + k_x)^2]} \sin 2k_z z \, e^{\mp Px} \tag{4.31}$$

with a corresponding acoustic velocity potential of

$$\underline{\Phi}_1[Res_2(s_2)] = \frac{4k_z^2}{Pa[1 + (\frac{iP}{a}) + k_x)^2]} \sin 2k_z z \, e^{i\omega t} \, e^{\mp Px} \tag{4.32}$$

This represents two standing disturbances decaying with x and negligible at great distances from the origin.

The residue of f(s) at s_g will be computed using (3.21) and is

$$H(s_g) \, e^{\pm ixs_g} \, Res\{G(s_g)\} \tag{4.33}$$

or

$$- \frac{\sin \sqrt{B+iC} \;\; z}{\sin \sqrt{B+iC} \;\; \ell_z} \, e^{\mp ax} \, e^{\mp k_x x} \, i \, \ell_z \tag{4.34}$$

where:

$$B = k_z^2 + k_x^2(1 - a^2) + a^2$$

$$C = 2a^2 k_x \tag{4.35}$$

The corresponding acoustic velocity potential is:

$$\Phi_1[\mathrm{Res}_g(s_g)] = k_z\, z \frac{\sin\sqrt{B+iC}\;\; z}{\sin\sqrt{B+iC}\;\ell_z}\; e^{\mp ax}\; e^{i(\omega t \mp k_x x)} \tag{4.36}$$

This corresponds to two traveling-decaying waves (both in x and z directions) going in opposite directions from x=0. At large distances from the origin its effect will be practically nil.

Combining results, only valid at $x >> 0$ or $x << 0$, we have:

$$\Phi = \Phi_0 + \varepsilon\, \Phi_1$$

$$= \sin k_z z\ [\cos(\omega t - k_x x) - \varepsilon \frac{2k_z^{\,2}}{k_x} \frac{a}{a^2 + k_x^{\,2}(a-1)^2} \sin(\omega t \mp k_x)] \tag{4.37}$$

It can be noticed, again, that the traveling disturbance is 90° out of phase with the original propagating wave.

V. RESULTS AND CONCLUSIONS

The results obtained show that the effects of very small irregularities in the physical boundaries of the duct can have appreciable importance in the waveduct sound propagation even at very large distances from the disturbances. In the case of the sinusoidal wall, resonances can exist, which could have the same order of magnitude of the original propagating wave. These resonances do not appear in the two other special cases studied, the δ-function disturbance and the decaying wall. It seems that the theory works and can be applied to other types of corrugations, once the Fourier transformation of the function describing the wall disturbance is known. However, complications can arise depending on the complexity of the above mentioned Fourier transform, making the solution integral difficult to evaluate.

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